

FITTED AND UNFITTED DOMAIN DECOMPOSITION USING PENALTY FREE NITSCHE METHOD FOR THE POISSON PROBLEM WITH DISCONTINUOUS MATERIAL PARAMETERS

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Abstract. In this paper, we study the stability of the non symmetric version of the Nitsche's method without penalty for domain decomposition. The Poisson problem is considered as a model problem. The computational domain is divided into two subdomain that can have different material parameters. In the first half of the paper we are interested in nonconforming domain decomposition, each subdomain is meshed independently of each other. In the second half, we study unfitted domain decomposition, the computational domain has only one mesh and we allow the interface to cut elements of the mesh. The fictitious domain method is used to handle this specificity. We prove H^1 -convergence and L^2 -convergence of the error in both cases. Some numerical results are provided to corroborate the theoretical study.

Key words. Nitsche's method; interface problem; Poisson problem; nonconforming domain decomposition, unfitted domain decomposition, fictitious domain.

1. Introduction. The two main methods that can be used for the weak enforcement of the boundary and/or interface conditions are the Lagrange multipliers method and the Nitsche's method that is a penalty based method. The Nitsche's method that has been introduced in 1971 [21] is known to have a symmetric and a nonsymmetric version [12, 18]. In this work we consider a nonsymmetric penalty free Nitsche's method [9, 4], this method can be seen as a Lagrange multiplier method, where the Lagrange multiplier has been replaced by the boundary fluxes of the discrete elliptic operator. The method does not have any additional degrees of freedom nor penalty parameter.

The Nitsche's method has been applied to nonconforming domain decomposition with its symmetric and nonsymmetric version by Becker et al. [3] for the Poisson problem. The method has been extended by Burman and Zunino [10] using a weighted average of the fluxes at the interface for the advection-diffusion-reaction problem. Several difficulties in the analysis can be handled by taking the right choice of weights (see [11, 2, 10]). For unfitted domain decomposition [15, 2] the interface can cut elements of the mesh, we handle this problem using the fictitious domain approach [13, 14, 1, 16, 6, 9].

In the second section of this paper, we study the nonconforming domain decomposition where each subdomain are meshed independently. In the third section, we extend the results to unfitted domain decomposition using the fictitious domain method, the fourth section shows a few numerical examples to illustrate the theoretical study.

Let Ω_1 and Ω_2 be two convex bounded domain in \mathbb{R}^2 with polygonal boundary, these two domains share an interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. We define the domain $\Omega = \Omega_1 \cup \Omega_2$ with boundary $\partial\Omega$, an example of Ω is represented in Figure 1. The Poisson problem

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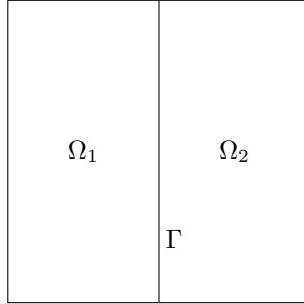


FIG. 1. Example of computational domain Ω .

considered can be expressed as

$$\begin{aligned} -\mu_i \Delta u^i &= f_i && \text{in } \Omega_i, \quad i = 1, 2, \\ u^i &= 0 && \text{on } \partial\Omega \cap \Omega_i, \quad i = 1, 2, \\ u^1 - u^2 &= 0 && \text{on } \Gamma, \\ \mu_1 \nabla u^1 \cdot n_1 + \mu_2 \nabla u^2 \cdot n_2 &= 0 && \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where u^i and μ_i are respectively the unknown and the diffusivity in Ω_i , $f_i \in L^2(\Omega_i)$ is a given body force. In this paper C will be used as a generic positive constant that may change at each occurrence, we will use the notation $a \lesssim b$ for $a \leq Cb$. For simplicity we will write the L^2 -norm on a domain Θ , $\|\cdot\|_{L^2(\Theta)}$ as $\|\cdot\|_\Theta$.

2. Fitted domain decomposition.

2.1. Preliminaries. The set $\{\mathcal{T}_h^i\}_h$ defines the family of quasi-uniform and shape regular triangulations fitted to Ω_i . We define the shape regularity as the existence of a constant $c_\rho \in \mathbb{R}_+$ for the family of triangulations such that, with ρ_K the radius of the largest circle inscribed in an element K , there holds

$$\frac{h_K}{\rho_K} \leq c_\rho \quad \forall K \in \mathcal{T}_h^i, \quad i = 1, 2.$$

In a generic sense we define K as a triangle in a triangulation \mathcal{T}_h^i and $h_K := \text{diam}(K)$ is the diameter of K . Then we define $h_i := \max_{K \in \mathcal{T}_h^i} h_K$ as the mesh parameter for a given triangulation \mathcal{T}_h^i . We study the domain decomposition problem with two subdomains that can be meshed independently, we make the assumption $\mu_2 h_1 \geq \mu_1 h_2$ and we set $h := \max(h_1, h_2)$.

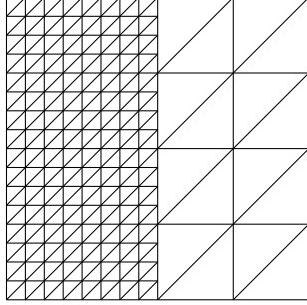
Let $V_i := \{v \in H^1(\Omega_i) : v|_{\partial\Omega} = 0\}$ for $i = 1, 2$, then $u = (u^1, u^2) \in V_1 \times V_2$. $\mathbb{P}_k(K)$ defines the space of polynomials of degree less than or equal to k on the element K . On each domain Ω_i we define the space of continuous piecewise polynomial functions

$$V_i^{kh} := \{u_h \in V_i : u_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h^i\}, \quad k \geq 1,$$

$V_h^k = V_1^{kh} \times V_2^{kh}$, every function in V_h^k has two components $v_h = (v_h^1, v_h^2)$. We now recall two classical inequalities.

LEMMA 2.1. *There exists $C_T \in \mathbb{R}_+$ such that for all $w \in H^1(K)$ and for all $K \in \mathcal{T}_h$, the trace inequality holds*

$$\|w\|_{\partial K} \leq C_T \left(h_K^{-\frac{1}{2}} \|w\|_K + h_K^{\frac{1}{2}} \|\nabla w\|_K \right).$$

FIG. 2. Example of mesh of Ω .

LEMMA 2.2. There exists $C_I \in \mathbb{R}_+$ such that for all $u_h \in \mathbb{P}_k(K)$ and for all $K \in \mathcal{T}_h$, the **inverse inequality** holds

$$\|\nabla u_h\|_K \leq C_I h_K^{-1} \|u_h\|_K.$$

At the interface we use the notations

$$[w] = w^1 - w^2,$$

for the jump, and

$$\{w\} = \omega_1 w^1 + \omega_2 w^2 \quad , \quad \langle w \rangle = \omega_2 w^1 + \omega_1 w^2,$$

with the following weights

$$\omega_1 = \frac{h_1 \mu_2}{h_1 \mu_2 + h_2 \mu_1} \quad , \quad \omega_2 = \frac{h_2 \mu_1}{h_1 \mu_2 + h_2 \mu_1}.$$

At the interface Γ we define the normal vector $n := n_1 = -n_2$, then we define

$$\{\mu \nabla w \cdot n\} = \omega_1 \mu_1 \nabla w_1 \cdot n + \omega_2 \mu_2 \nabla w_2 \cdot n,$$

we note that

$$\{\mu \nabla u \cdot n\} = \mu_1 \nabla u^1 \cdot n_1 = -\mu_2 \nabla u^2 \cdot n_2.$$

To simplify the notations in the analysis we set

$$\gamma = \frac{\mu_1 \mu_2}{h_1 \mu_2 + h_2 \mu_1}.$$

We now introduce a structure of patches that will be used in the upcoming inf-sup analysis similarly as [7, 4]. Let the interface elements be the triangles with either a face or a vertex on the interface Γ . We regroup the interface elements of Ω_i in closed disjoint patches P_j^i with boundary ∂P_j^i , $j = 1, \dots, N_P^i$. N_P^i defines the total number of patches. Note that $P^i := \cup_j P_j^i$. Let $F_j^i := \partial P_j^i \cap \Gamma$. For each F_j^i there exists two positive constants c_1, c_2 such that for all j

$$c_1 h_i \leq \text{meas}(F_j^i) \leq c_2 h_i,$$

with h_i the meshsize of the subdomain considered. Let us focus on the patches P_j^1 attached to the domain Ω_1 . Each patch P_j^1 is associated with a function $\phi_j \in V_1^{1h}$ defined such that for each node $x_i \in \mathcal{T}_h^1$ of the face F_j^1 let

$$\phi_j(x_i) = \begin{cases} 0 & \text{for } x_i \in \Omega_1 \setminus \dot{F}_j^1 \\ 1 & \text{for } x_i \in \dot{F}_j^1, \end{cases} \quad (2.1)$$

with $i = 1, \dots, N_n$. N_n is the number of node in the triangulation \mathcal{T}_h^1 and \dot{F}_j^1 defines the interior of the face F_j^1 . We define the function $v_\Gamma^1 \in V_1^{kh}$ such that

$$v_\Gamma^1 = \alpha \sum_{j=1}^{N_P^1} v_j^1, \quad (2.2)$$

with

$$v_j^1 = \nu_j \phi_j, \quad , \quad \nu_j \in \mathbb{R}. \quad (2.3)$$

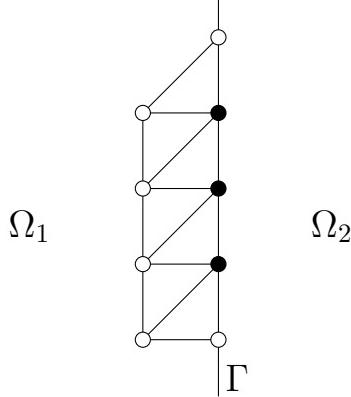


FIG. 3. Example Patch P_j^1 on the interface elements, the function ϕ_j is equal to 0 in the nonfilled nodes, 1 in the filled nodes.

In order to define the properties of v_j^1 , we define the P_0 -projection of a function w on an interval I

$$\bar{w}^I := \text{meas}(I)^{-1} \int_I w \, ds.$$

We may now define on each face F_j^1 the function v_j^1 such that

$$\text{meas}(F_j^1)^{-1} \int_{F_j^1} \nabla v_j^1 \cdot n \, ds = h_1^{-1} \overline{\llbracket u_h \rrbracket}_{F_j^1}. \quad (2.4)$$

Applying the Poincaré inequality, on each patch P_j^1 the function v_j^1 has the following property

$$\|v_j^1\|_{P_j^1} \lesssim h_1 \|\nabla v_j^1\|_{P_j^1}. \quad (2.5)$$

It is straightforward to observe that on each face F_j^i it holds

$$\left\| u_h^i - \overline{u}_h^{F_j^i} \right\|_{F_j^i} \lesssim h_i \|\nabla u_h^i\|_{F_j^i}. \quad (2.6)$$

The Lemma 4.1 of [7] allows us to write the following inequality for each patch P_j^1 of the triangulation \mathcal{T}_h^1 ,

$$\|\nabla v_j^1\|_{P_j^1}^2 \lesssim \left\| h_1^{-\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1}^2. \quad (2.7)$$

LEMMA 2.3. *Considering the patches P_j^i as defined above $\forall u_h \in V_h^k$ the following inequality holds*

$$\sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} \geq \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket \right\|_{F_j^1} - C\omega_1 \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1} - C\omega_2 \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}.$$

Proof. Considering the triangle inequality and the definition of the jump we can write

$$\begin{aligned} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket \right\|_{F_j^1} &\leq \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} + \left\| \gamma^{\frac{1}{2}} \left(\llbracket u_h \rrbracket - \llbracket u_h \rrbracket^{F_j^1} \right) \right\|_{F_j^1} \\ &\leq \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} + \left\| \gamma^{\frac{1}{2}} \left(u_h^1 - u_h^2 - \left(\overline{u}_h^{F_j^1} - \overline{u}_h^{F_j^1} \right) \right) \right\|_{F_j^1}. \end{aligned}$$

Taking the sum over the whole interface and using the triangle inequality once again followed by inequality (2.6), trace inequality of Lemma 2.1 and inverse inequality of Lemma 2.2 we obtain

$$\begin{aligned} \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket \right\|_{F_j^1} &\leq \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} + \gamma^{\frac{1}{2}} \sum_{j=1}^{N_p^1} \left\| u_h^1 - \overline{u}_h^{F_j^1} \right\|_{F_j^1} + \gamma^{\frac{1}{2}} \sum_{j=1}^{N_p^2} \left\| u_h^2 - \overline{u}_h^{F_j^2} \right\|_{F_j^2} \\ &\leq \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} + \gamma^{\frac{1}{2}} h_1 \sum_{j=1}^{N_p^1} \|\nabla u_h^1\|_{F_j^1} + \gamma^{\frac{1}{2}} h_2 \sum_{j=1}^{N_p^2} \|\nabla u_h^2\|_{F_j^2} \\ &\leq \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{F_j^1} \right\|_{F_j^1} + C\omega_1 \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1} + C\omega_2 \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}. \quad \square \end{aligned}$$

We end this section by defining the triple norm of a function w

$$\|w\|^2 = \sum_{i=1}^2 \left\| \mu_i^{\frac{1}{2}} \nabla w^i \right\|_{\Omega_i}^2 + \left\| \gamma^{\frac{1}{2}} \llbracket w \rrbracket \right\|_{\Gamma}^2. \quad (2.8)$$

2.2. Finite element formulation. Classically for the Poisson problem (1.1) for each subdomain domain Ω_i we obtain by integration by parts

$$(\mu_i \nabla u^i, \nabla v^i)_{\Omega_i} - \langle \mu_i \nabla u^i \cdot n_i, v^i \rangle_{\Gamma} = (f, v^i)_{\Omega_i}, \quad \forall v^i \in V_i.$$

By taking the sum of the boundary terms we obtain

$$\sum_{i=1}^2 \langle \mu_i \nabla u^i \cdot n_i, v^i \rangle_\Gamma = \int_\Gamma \llbracket (\mu \nabla u \cdot n) v \rrbracket \, ds = \langle \{ \mu \nabla u \cdot n \}, \llbracket v \rrbracket \rangle_\Gamma + \langle \llbracket \mu \nabla u \cdot n \rrbracket, \langle v \rangle \rangle_\Gamma,$$

(1.1) tells us that $\llbracket \mu \nabla u \cdot n \rrbracket = 0$, then we get

$$\sum_{i=1}^2 (\mu_i \nabla u^i, \nabla v^i)_{\Omega_i} - \langle \{ \mu \nabla u \cdot n \}, \llbracket v \rrbracket \rangle_\Gamma = \sum_{i=1}^2 (f_i, v^i)_{\Omega_i}.$$

Adding the Nitsche term, it leads to the following finite element formulation : Find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h^k, \quad (2.9)$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{i=1}^2 (\mu_i \nabla u_h^i, \nabla v_h^i)_{\Omega_i} - \langle \{ \mu \nabla u_h \cdot n \}, \llbracket v_h \rrbracket \rangle_\Gamma + \langle \{ \mu \nabla v_h \cdot n \}, \llbracket u_h \rrbracket \rangle_\Gamma, \\ L_h(v_h) &= \sum_{i=1}^2 (f_i, v_h^i)_{\Omega_i}. \end{aligned}$$

2.3. Inf-sup stability. This section leads to the inf-sup stability of the penalty free scheme previously introduced, we first prove an auxiliary Lemma.

LEMMA 2.4. *For $u_h, v_h \in V_h^k$ with $v_h = u_h + v_\Gamma^1$, v_Γ^1 defined by equations (2.2) and (2.3), there exists a positive constant β_0 such that the following inequality holds*

$$\beta_0 \|u_h\|^2 \leq A_h(u_h, v_h).$$

Proof. Using the definition of v_Γ^1 , we can write the following

$$A_h(u_h, v_h) = A_h(u_h, u_h) + \alpha \sum_{j=1}^{N_p^1} A_h(u_h, v_j^1).$$

Clearly we have

$$A_h(u_h, u_h) = \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{\Omega_1}^2 + \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{\Omega_2}^2,$$

and

$$A_h(u_h, v_j^1) = (\mu_1 \nabla u_h^1, \nabla v_j^1)_{P_j^1} - \langle \{ \mu \nabla u_h \cdot n \}, v_j^1 \rangle_{F_j^1} + \omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, \llbracket u_h \rrbracket \rangle_{F_j^1}.$$

Using Cauchy-Schwarz inequality and inequality (2.7)

$$\begin{aligned} (\mu_1 \nabla u_h^1, \alpha \nabla v_j^1)_{P_j^1} &\geq - \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1} \alpha \mu_1^{\frac{1}{2}} \| \nabla v_j^1 \|_{P_j^1} \\ &\geq -\epsilon \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 - \frac{C\alpha^2}{4\epsilon} \left(1 + \frac{h_2 \mu_1}{h_1 \mu_2} \right) \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket_{F_j^1} \right\|_{F_j^1}^2. \end{aligned}$$

Using the trace and inverse inequalities, (2.5) and (2.7) we can write

$$\begin{aligned}
& \left\langle \{\mu \nabla u_h \cdot n\}, \alpha v_j^1 \right\rangle_{F_j^1} = \left\langle \omega_1 \mu_1 \nabla u_h^1 \cdot n + \omega_2 \mu_2 \nabla u_h^2 \cdot n, \alpha v_j^1 \right\rangle_{F_j^1} \\
&= \left\langle (\omega_1 \mu_1 h_1)^{\frac{1}{2}} \nabla u_h^1 \cdot n + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \nabla u_h^2 \cdot n, \alpha \gamma^{\frac{1}{2}} v_j^1 \right\rangle_{F_j^1} \\
&\leq \left((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\nabla u_h^1 \cdot n\|_{F_j^1} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\nabla u_h^2 \cdot n\|_{F_j^1} \right) \alpha \left\| \gamma^{\frac{1}{2}} v_j^1 \right\|_{F_j^1} \\
&\lesssim \left((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\nabla u_h^1 \cdot n\|_{F_j^1} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\nabla u_h^2 \cdot n\|_{F_j^1} \right) \alpha \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1}.
\end{aligned}$$

Taking the sum over the full boundary Γ and using trace and inverse inequalities once again we obtain

$$\begin{aligned}
\sum_{j=1}^{N_p^1} \left\langle \{\mu \nabla u_h \cdot n\}, \alpha v_j^1 \right\rangle_{F_j^1} &\leq \frac{C\alpha^2}{2\epsilon} \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon (\omega_1 \mu_1 h_1) \sum_{j=1}^{N_p^1} \|\nabla u_h^1 \cdot n\|_{F_j^1}^2 \\
&\quad + \epsilon (\omega_2 \mu_2 h_2) \sum_{j=1}^{N_p^2} \|\nabla u_h^2 \cdot n\|_{F_j^2}^2 \\
&\leq \frac{C\alpha^2}{2\epsilon} \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \omega_1 \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 \\
&\quad + \epsilon \omega_2 \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2.
\end{aligned}$$

Using the property (2.4) of v_j^1 we can write for each face F_j^1

$$\alpha \omega_1 \left\langle \mu_1 \nabla v_j^1 \cdot n, [u_h] \right\rangle_{F_j^1} \geq \alpha \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1}^2 + \alpha \omega_1 \left\langle \mu_1 \nabla v_j^1 \cdot n, [u_h] - \overline{[u_h]}^{F_j^1} \right\rangle_{F_j^1}.$$

Using the trace inequality and inequality (2.7) we get

$$\begin{aligned}
& \alpha \omega_1 \left\langle \mu_1 \nabla v_j^1 \cdot n, [u_h] - \overline{[u_h]}^{F_j^1} \right\rangle_{F_j^1} \\
&\leq \alpha \omega_1 \mu_1 \|\nabla v_j^1 \cdot n\|_{F_j^1} \left\| [u_h] - \overline{[u_h]}^{F_j^1} \right\|_{F_j^1} \\
&\lesssim \alpha \omega_1 \mu_1 h_1^{-\frac{1}{2}} \|\nabla v_j^1\|_{P_j^1} \left\| u_h^1 - u_h^2 - \left(\overline{u_h^1}^{F_j^1} - \overline{u_h^2}^{F_j^1} \right) \right\|_{F_j^1} \\
&\lesssim \alpha \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1} \left(\frac{\omega_1 \mu_1}{h_1} \right)^{\frac{1}{2}} \left\| \left(u_h^1 - \overline{u_h^1}^{F_j^1} \right) - \left(u_h^2 - \overline{u_h^2}^{F_j^1} \right) \right\|_{F_j^1} \\
&\lesssim \alpha \left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1} \left(\frac{\omega_1 \mu_1}{h_1} \right)^{\frac{1}{2}} \left(\left\| u_h^1 - \overline{u_h^1}^{F_j^1} \right\|_{F_j^1} + \left\| u_h^2 - \overline{u_h^2}^{F_j^1} \right\|_{F_j^1} \right).
\end{aligned}$$

Taking the sum over the whole interface Γ and using Young's inequality and (2.6)

$$\begin{aligned}
& \sum_{j=1}^{N_p^1} \alpha \omega_1 \left\langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1} \right\rangle_{F_j^1} \\
& \leq \sum_{j=1}^{N_p^1} \left[\frac{C\alpha^2}{2\epsilon} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \frac{\omega_1 \mu_1}{h_1} \left\| u_h^1 - \overline{u_h^1}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \frac{\omega_1 \mu_1}{h_1} \left\| u_h^2 - \overline{u_h^2}^{F_j^1} \right\|_{F_j^1}^2 \right] \\
& \leq \frac{C\alpha^2}{2\epsilon} \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \frac{\omega_1 \mu_1}{h_1} \sum_{j=1}^{N_p^1} \left\| u_h^1 - \overline{u_h^1}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \frac{\omega_1 \mu_1}{h_1} \sum_{j=1}^{N_p^2} \left\| u_h^2 - \overline{u_h^2}^{F_j^2} \right\|_{F_j^2}^2 \\
& \leq \frac{C\alpha^2}{2\epsilon} \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2 + \epsilon \omega_1 \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 + \epsilon \omega_2 \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2.
\end{aligned}$$

It allows us to write

$$\begin{aligned}
\sum_{j=1}^{N_p^1} \alpha \omega_1 \left\langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!] \right\rangle_{F_j^1} & \geq -\epsilon \omega_1 \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 - \epsilon \omega_2 \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2 \\
& + \alpha \left(1 - \frac{C\alpha}{2\epsilon} \right) \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2.
\end{aligned}$$

The full bilinear form A_h now has the following lowerbound

$$\begin{aligned}
A_h(u_h, v_h) & \geq \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{\Omega_1 \setminus P^1}^2 + \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{\Omega_2 \setminus P^2}^2 + C_a \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 \\
& + C_b \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2 + C_c \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2,
\end{aligned}$$

with the constants

$$\begin{aligned}
C_a & = 1 - \epsilon (2\omega_1 + 1), \\
C_b & = 1 - 2\epsilon\omega_2, \\
C_c & = \alpha \left(1 - \alpha \frac{C}{4\epsilon} \left(5 + \frac{h_2 \mu_1}{h_1 \mu_2} \right) \right).
\end{aligned}$$

Using Lemma 2.3 it becomes

$$\begin{aligned}
A_h(u_h, v_h) & \geq \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{\Omega_1 \setminus P^1}^2 + \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{\Omega_2 \setminus P^2}^2 + (C_a - \omega_1 C C_c) \sum_{j=1}^{N_p^1} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 \\
& + (C_b - \omega_2 C C_c) \sum_{j=1}^{N_p^2} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2 + C_c \sum_{j=1}^{N_p^1} \left\| \gamma^{\frac{1}{2}} \overline{[\![u_h]\!]}^{F_j^1} \right\|_{F_j^1}^2.
\end{aligned}$$

First we fix $\epsilon = \min \left[\frac{1}{2(2\omega_1+1)}, \frac{1}{4\omega_2} \right]$. The constant C_c will be positive for

$$\alpha < \frac{4\epsilon\omega_1}{C(4\omega_1+1)}.$$

The terms $(C_a - \omega_1 CC_c)$ and $(C_b - \omega_2 CC_c)$ will be both positive for

$$\alpha < \frac{1}{2C}.$$

$\mathcal{O}(\epsilon) = \min [\mathcal{O}(1), \mathcal{O}(\omega_2^{-1})]$, $\mathcal{O}(\alpha) = \min [\mathcal{O}(\epsilon\omega_1), \mathcal{O}(1)]$ and $\mathcal{O}(\beta_0) = \mathcal{O}(\alpha)$. \square

THEOREM 2.5. *There exists a positive constant $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta \|u_h\| \leq \sup_{v_h \in V_h^k} \frac{A_h(u_h, v_h)}{\|v_h\|}.$$

Proof. Considering Lemma 2.4, the only thing that we need to show is

$$\|v_h\| \lesssim \|u_h\|.$$

The triangle inequality gives

$$\|v_h\| = \|u_h\| + \sum_{j=1}^{N_p^1} \|v_j^1\|.$$

The triple norm (2.8) gives

$$\|v_j^1\|^2 = \|\mu_1^{\frac{1}{2}} \nabla v_j^1\|_{P_j^1}^2 + \|\gamma^{\frac{1}{2}} v_j^1\|_{F_j^1}^2.$$

Recalling the inequality (2.7) and

$$\left\| \gamma^{\frac{1}{2}} \overline{[u_h]}^{F_j^1} \right\|_{F_j^1} \lesssim \left\| \gamma^{\frac{1}{2}} [u_h] \right\|_{F_j^1} \lesssim \|u_h\|,$$

it gives the appropriate upper bound

$$\left\| \mu_1^{\frac{1}{2}} \nabla v_j^1 \right\|_{P_j^1} \lesssim \omega_1^{-\frac{1}{2}} \|u_h\|.$$

Using the trace inequality of Lemma 2.1 and the inequality (2.5)

$$\left\| \gamma^{\frac{1}{2}} v_j^1 \right\|_{F_j^1} \lesssim \omega_1^{\frac{1}{2}} \left\| \mu_1^{\frac{1}{2}} \nabla v_j^1 \right\|_{P_j^1} \lesssim \|u_h\|.$$

Note that $\mathcal{O}(\beta) = \mathcal{O}(\beta_0 \omega_1^{\frac{1}{2}})$. \square

2.4. A priori error estimate. The proof of the stability done in the previous part leads to the study of the a priori error estimate in the triple norm, the following consistency relation characterizes the Galerkin orthogonality.

LEMMA 2.6. *If $u \in H^2(\Omega_1) \times H^2(\Omega_2)$ is the solution of (1.1) and $u_h \in V_h^k$ the solution of (2.9) the following property holds*

$$A_h(u - u_h, v_h) = 0 \quad , \quad \forall v_h \in V_h^k.$$

Proof. We observe that $A_h(u, v_h) = L_h(v_h) = A_h(u_h, v_h)$, $\forall v_h \in V_h^k$. \square

In order to study the a priori error estimate, we introduce an auxiliary norm

$$\|w\|_\chi = |\!|\!| w |\!|\!| + \left\| \mu_1^{\frac{1}{2}} h_1^{\frac{1}{2}} \nabla w^1 \cdot n \right\|_\Gamma + \left\| \mu_2^{\frac{1}{2}} h_2^{\frac{1}{2}} \nabla w^2 \cdot n \right\|_\Gamma.$$

LEMMA 2.7. *Let $w \in H^2(\Omega_1) \times H^2(\Omega_2) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant M such that the bilinear form $A_h(\cdot, \cdot)$ has the property*

$$A_h(w, v_h) \leq M \|w\|_\chi |\!|v_h|\!|.$$

Proof. Using the trace inequality of Lemma 2.1 and the Cauchy Schwartz inequality we can show

$$\begin{aligned} \langle \{\mu \nabla w \cdot n\}, [\![v_h]\!] \rangle_\Gamma &= \left\langle (\omega_1 \mu_1 h_1)^{\frac{1}{2}} \nabla w \cdot n + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \nabla w \cdot n, \gamma^{\frac{1}{2}} [\![v_h]\!] \right\rangle_\Gamma \\ &\leq \left(\omega_1^{\frac{1}{2}} \left\| \mu_1^{\frac{1}{2}} h_1^{\frac{1}{2}} \nabla w^1 \cdot n \right\|_\Gamma + \omega_2^{\frac{1}{2}} \left\| \mu_2^{\frac{1}{2}} h_2^{\frac{1}{2}} \nabla w^2 \cdot n \right\|_\Gamma \right) \left\| \gamma^{\frac{1}{2}} [\![v_h]\!] \right\|_\Gamma, \\ \langle \{\mu \nabla v_h \cdot n\}, [\![w]\!] \rangle_\Gamma &\leq \left(\omega_1^{\frac{1}{2}} \left\| \mu_1^{\frac{1}{2}} h_1^{\frac{1}{2}} \nabla v_h^1 \cdot n \right\|_\Gamma + \omega_2^{\frac{1}{2}} \left\| \mu_2^{\frac{1}{2}} h_2^{\frac{1}{2}} \nabla v_h^2 \cdot n \right\|_\Gamma \right) \left\| \gamma^{\frac{1}{2}} [\![w]\!] \right\|_\Gamma \\ &\lesssim \left(\omega_1^{\frac{1}{2}} \left\| \mu_1^{\frac{1}{2}} \nabla v_h^1 \right\|_{\Omega_1} + \omega_2^{\frac{1}{2}} \left\| \mu_2^{\frac{1}{2}} \nabla v_h^2 \right\|_{\Omega_2} \right) \left\| \gamma^{\frac{1}{2}} [\![w]\!] \right\|_\Gamma. \end{aligned}$$

Using these two upper bound it is straightforward to show that

$$\sum_{i=1}^2 \left(\mu_i \nabla w^i, \nabla v_h^i \right)_{\Omega_i} + \langle \{\mu \nabla w \cdot n\}, [\![v_h]\!] \rangle_\Gamma + \langle \{\mu \nabla v_h \cdot n\}, [\![w]\!] \rangle_\Gamma \lesssim \|w\|_\chi |\!|v_h|\!|.$$

$$\mathcal{O}(M) = \min \left[\mathcal{O}\left(\omega_1^{\frac{1}{2}}\right), \mathcal{O}\left(\omega_2^{\frac{1}{2}}\right) \right]. \quad \square$$

PROPOSITION 2.8. *If $u \in H^{k+1}(\Omega_1) \times H^{k+1}(\Omega_2)$ is the solution of (1.1) and $u_h \in V_h^k$ the solution of (2.9), then there holds*

$$|\!|u - u_h|\!| \leq C_\mu h^k |u|_{H^{k+1}(\Omega)},$$

with C_μ a positive constant that depends on μ and the mesh geometry.

Proof. Let π_h^k denote the nodal interpolant, the approximation properties for each $K \in \mathcal{T}_h^i$ with $i = 1, 2$, give

$$\|u - \pi_h^k u\|_K + h_K \|\nabla(u - \pi_h^k u)\|_K + h_K^2 \|D^2(u - \pi_h^k u)\|_K \lesssim h_K^{k+1} |u|_{H^{k+1}(K)}.$$

Using this property and the trace inequality it is straightforward to show that

$$|\!|u - \pi_h^k u|\!| \lesssim \|u - \pi_h^k u\|_\chi \lesssim \mu_1^{\frac{1}{2}} h_1^k |u^1|_{H^{k+1}(\Omega_1)} + \mu_2^{\frac{1}{2}} h_2^k |u^2|_{H^{k+1}(\Omega_2)}.$$

Using the Galerkin orthogonality of Lemma 2.6, the Theorem 2.5 and the Lemma 2.7 we can write

$$\beta |\!|u_h - \pi_h^k u|\!| \leq \frac{A_h(u - \pi_h^k u, v_h)}{|\!|v_h|\!|} \leq M \|u - \pi_h^k u\|_\chi.$$

Using this property and the triangle inequality we obtain

$$\|u - u_h\| \leq \|u - \pi_h^k u\| + \frac{M}{\beta} \|\pi_h^k u\|_X.$$

The result follows. $\mathcal{O}(C_\mu) = \mathcal{O}\left(\left(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}\right)\left(1 + \frac{M}{\beta}\right)\right)$. \square

LEMMA 2.9. *Let $u \in H^{k+1}(\Omega_1) \times H^{k+1}(\Omega_2)$ be the solution of (1.1) and u_h the solution of (2.9), then there holds*

$$\|u - u_h\|_\Omega \leq C'_\mu h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)},$$

with C'_μ is a positive constant that depends on μ and the mesh geometry.

Proof. Let z satisfy the adjoint problem

$$\begin{aligned} -\mu_i \Delta z^i &= u^i - u_h^i && \text{in } \Omega_i, \quad i = 1, 2, \\ z^i &= 0 && \text{on } \partial\Omega \cap \Omega_i, \quad i = 1, 2, \\ z^1 - z^2 &= 0 && \text{on } \Gamma, \\ \mu_1 \nabla z^1 \cdot n_1 + \mu_2 \nabla z^2 \cdot n_2 &= 0 && \text{on } \Gamma, \end{aligned}$$

Then we can write

$$\begin{aligned} \|u^i - u_h^i\|_{\Omega_i}^2 &= (u^i - u_h^i, -\mu_i \Delta z^i)_{\Omega_i} \\ &= (\nabla(u^i - u_h^i), \mu_i \nabla z^i)_{\Omega_i} - \langle \mu_i \nabla z^i \cdot n_i, u^i - u_h^i \rangle_\Gamma, \end{aligned}$$

it leads to

$$\begin{aligned} \|u - u_h\|_\Omega^2 &\leq \|u^1 - u_h^1\|_{\Omega_1}^2 + \|u^2 - u_h^2\|_{\Omega_2}^2, \\ &\leq \sum_{i=1}^2 (\nabla(u^i - u_h^i), \mu_i \nabla z^i) - \langle \{\mu \nabla z \cdot n\}, [u - u_h] \rangle_\Gamma, \\ &\leq A_h(u - u_h, z) - 2 \langle \{\mu \nabla z \cdot n\}, [u - u_h] \rangle_\Gamma. \end{aligned}$$

Using the global trace inequality $\|\nabla z^i \cdot n\|_\Gamma \lesssim \|z^i\|_{H^2(\Omega_i)}$ for $i = 1, 2$, we can write

$$\begin{aligned} |\langle \{\mu \nabla z \cdot n\}, [u - u_h] \rangle_\Gamma| &\leq \left((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\nabla z^1 \cdot n\|_\Gamma + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\nabla z^2 \cdot n\|_\Gamma \right) \|\gamma^{\frac{1}{2}} [u - u_h]\|_\Gamma \\ &\lesssim \left((\omega_1 \mu_1)^{\frac{1}{2}} + (\omega_2 \mu_2)^{\frac{1}{2}} \right) h^{\frac{1}{2}} \|z\|_{H^2(\Omega)} \|u - u_h\|. \end{aligned}$$

Using Lemma 2.6 and $(z^i - \pi_h^1 z^i)|_\Gamma \equiv 0$ for $i = 1, 2$

$$\begin{aligned} A_h(u - u_h, z) &= A_h(u - u_h, z - \pi_h^1 z), \\ &= \sum_{i=1}^2 (\nabla(u^i - u_h^i), \mu_i \nabla(z^i - \pi_h^1 z^i))_{\Omega_i} \\ &\quad + \langle \{\mu \nabla(z - \pi_h^1 z) \cdot n\}, [u - u_h] \rangle_\Gamma, \\ &\lesssim \left((\omega_1 \mu_1)^{\frac{1}{2}} + (\omega_2 \mu_2)^{\frac{1}{2}} \right) h \|z\|_{H^2(\Omega)} \|u - u_h\|. \end{aligned}$$

Then we get

$$\|u - u_h\|_{\Omega}^2 \lesssim C_{\mu} \left((\omega_1 \mu_1)^{\frac{1}{2}} + (\omega_2 \mu_2)^{\frac{1}{2}} \right) (h + h^{\frac{1}{2}}) h^k |u|_{H^{k+1}(\Omega)} \|z\|_{H^2(\Omega)}.$$

We conclude by applying the regularity estimate $\|z\|_{H^2(\Omega)} \lesssim \|u - u_h\|_{\Omega}$. $\mathcal{O}(C'_{\mu}) = \mathcal{O}\left(C_{\mu} \left((\omega_1 \mu_1)^{\frac{1}{2}} + (\omega_2 \mu_2)^{\frac{1}{2}} \right)\right)$. \square

3. Unfitted domain decomposition.

3.1. Preliminaries. In this section the two subdomains Ω_1 and Ω_2 are both meshed with only one triangulation. Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform and shape regular triangulation fitted to Ω , the shape regularity is defined in the same way as in the previous section. In a generic sense, F with normal n_F denotes a face of a triangle K in the triangulation \mathcal{T}_h . The mesh size is defined as $h := \max_{K \in \mathcal{T}_h} h_K$.

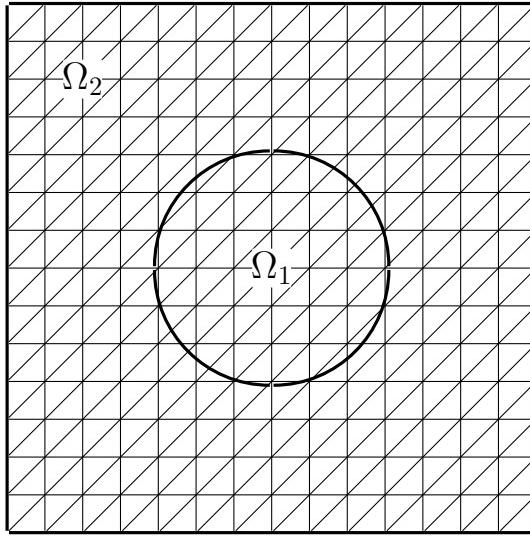


FIG. 4. Example of mesh of Ω .

Figure 4 shows an example of configuration, the mesh do not fit with the interface Γ . Let

$$\Omega_i^* := \{K \in \mathcal{T}_h \mid K \cap \Omega_i \neq \emptyset\},$$

We redefine the spaces defined in the previous section.

$$\begin{aligned} V_i &:= \{v \in H^1(\Omega_i^*) : v|_{\partial\Omega} = 0\}, \\ V_i^{kh} &:= \{u_h \in V_i : u_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\}, \end{aligned}$$

then $V_h^k := V_1^{kh} \times V_2^{kh}$. We define the set of elements that intersects the boundary

$$G_h := \{K \in \mathcal{T}_h \mid K \cap \Gamma \neq \emptyset\}.$$

For the sake of precision we make the following assumptions regarding the mesh \mathcal{T}_h and the boundary Γ :

- The boundary Γ intersects each element boundary ∂K exactly twice, and each (open) edge at most once for $K \in G_h$.
- Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between Γ and ∂K . We assume that Γ_K is a function of length on $\Gamma_{K,h}$; in local coordinates

$$\Gamma_{K,h} = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = 0\}$$

and

$$\Gamma_K = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

- We assume that for all $K \in G_h$ there exists $K' \notin G_h$ and $K \cap K' \neq \emptyset$ and such that the measures of K and K' are comparable in the sense that there exists $c_q > 1$ such that

$$c_q^{-1} \leq \frac{\text{meas}(K)}{\text{meas}(K')} \leq c_q$$

and that the faces F such that $K \cap F \neq \emptyset$ and $K' \cap F \neq \emptyset$ satisfy

$$|\text{meas}(K')| \leq c_q \text{meas}(F)^2.$$

- We assume that in a triangle K intersected by the interface Γ , the scalar product between the normal n_F of the face that does not intersects Γ and the normal of the interface n keeps the same sign in K .

We now extend the trace inequality for $v \in H^1(\Omega)$

$$\|v\|_{K \cap \Gamma} \lesssim \left(h_K^{-\frac{1}{2}} \|v\|_K + h_K^{\frac{1}{2}} \|\nabla v\|_K \right) \quad \forall K \in \mathcal{T}_h. \quad (3.1)$$

The inequality (3.1) has been shown in [15]. Let \mathbb{E}_i be an H^k -extension on Ω_i^* , $\mathbb{E}_i : H^k(\Omega_i) \rightarrow H^k(\Omega_i^*)$ such that $(\mathbb{E}_i w)|_{\Omega_i} = w$ and

$$\|\mathbb{E}_i w\|_{H^k(\Omega_i^*)} \lesssim \|w\|_{H^k(\Omega_i)} \quad , \quad k \geq 0. \quad (3.2)$$

Let π_h^* be the standard nodal interpolant, we construct the interpolation operator π_h such that $\pi_h w = (\pi_h^1 w, \pi_h^2 w)$ such that

$$\pi_h^i u^i := \pi_h^* \mathbb{E}_i u^i \quad (3.3)$$

We recall the interpolation estimates for $0 \leq r \leq s \leq k+1$,

$$\|u^i - \pi_h^* u^i\|_{H^r(K)} \lesssim h^{s-r} |u^i|_{H^s(K)} \quad \forall K \in \mathcal{T}_h, \quad (3.4)$$

$$\|u^i - \pi_h^* u^i\|_{H^r(F)} \lesssim h^{s-r-\frac{1}{2}} |u^i|_{H^s(K)} \quad \forall F \in \mathcal{F}. \quad (3.5)$$

Using the estimate (3.2) together with (3.4) and (3.5)

$$\|u^i - \pi_h^i u^i\|_{H^r(\Omega_i^*)} \lesssim h^{s-r} |u^i|_{H^s(\Omega_i)}, \quad (3.6)$$

$$\sum_{F \in \mathcal{F}} \|u^i - \pi_h^i u^i\|_{H^r(F)} \lesssim h^{s-r-\frac{1}{2}} |u^i|_{H^s(\Omega_i)}. \quad (3.7)$$

The stability analysis for this case is similar to the fitted case treated previously, first we need to adapt the structure of patches to this new configuration. Let us split the set G_h into N_p smaller disjoint set of elements G_j with $j = 1, \dots, N_p$. Let I_{G_j} be the set of nodes $\{x_n\}$ belonging to G_j . A generic node of the triangulation is designated by x_n , we define the sets of nodes I_j^1 and I_j^2 such that

$$I_j^1 := \{x_n \in I_{G_j} \mid x_n \in \Omega_1, x_n \notin I_{G_i} \forall i \neq j\},$$

$$I_j^2 := \{x_n \in I_{G_j} \mid x_n \in \Omega_2, x_n \notin I_{G_i} \forall i \neq j\},$$

now we define P_j^1 and P_j^2 for each G_j such that

$$P_j^1 := G_j \cup \{K \in \mathcal{T}_h \mid I_j^1 \in K\}, \quad P_j^2 := G_j \cup \{K \in \mathcal{T}_h \mid I_j^2 \in K\}.$$

Each patch P_j^i is constructed such that $I_j^i \neq \emptyset$ for $i = 1, 2$. Figure 3.1 shows an example of two patch P_j^1 and P_j^2 attached to a set of interface elements G_j . $\Gamma_j := \Gamma \cap G_j$ is the part of the boundary included in the patches P_j^1 and P_j^2 . For all j and $i = 1, 2$, the patch P_j^i has the following properties

$$h \lesssim \text{meas}(\Gamma_j) \lesssim h \quad \text{and} \quad h^2 \lesssim \text{meas}(P_j^i) \lesssim h^2. \quad (3.8)$$

The function ϕ_j attached to P_j^1 introduced in (2.1) is modified such that

$$\phi_j(x_i) = \begin{cases} 0 & \text{for } x_i \notin I_j^1 \\ 1 & \text{for } x_i \in I_j^1, \end{cases}$$

with $i = 1, \dots, N_n$. N_n is the number of nodes in the triangulation \mathcal{T}_h . Similarly as

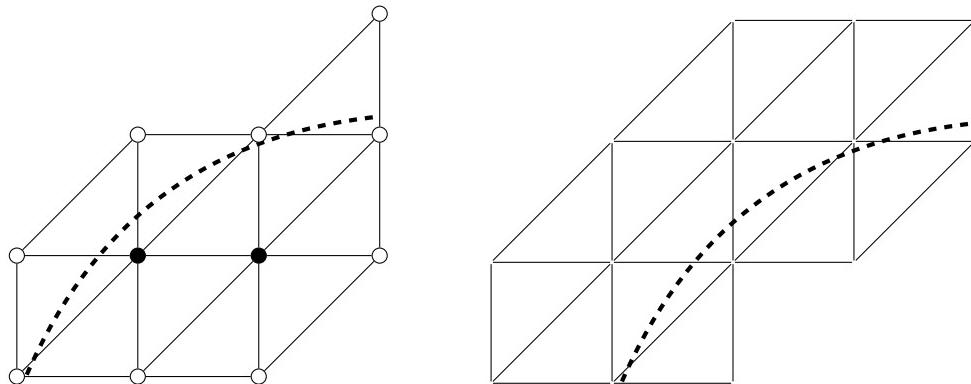


FIG. 5. Zoom of Figure 4. For a set G_j , left : example of P_j^1 , the function ϕ_j is equal to 0 in the nonfilled nodes, 1 in the filled nodes ; right : example of P_j^2 .

the fitted case we define the function $v_\Gamma^1 \in V_1^{kh}$ such that

$$v_\Gamma^1 = \alpha \sum_{j=1}^{N_p} v_j^1, \quad (3.9)$$

with

$$v_j^1 = \nu_j \phi_j, \quad \nu_j \in \mathbb{R}. \quad (3.10)$$

The function v_j^1 has the property

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla v_j^1 \cdot n \, ds = h^{-1} \overline{\llbracket u_h \rrbracket}_{\Gamma_j}. \quad (3.11)$$

It is straightforward to remark that (2.6) and (2.5) still hold for this new configuration

$$\|u_h^i - \overline{u_h^i}_{\Gamma_j}\|_{\Gamma_j} \lesssim h \|\nabla u_h^i\|_{\Gamma_j}, \quad (3.12)$$

$$\|v_j^1\|_{P_j^1} \lesssim h \|\nabla v_j^1\|_{P_j^1}. \quad (3.13)$$

The Lemma 1 of [5] combined with the regularity assumptions on the mesh made previously allows us to extend (2.7) to

$$\|\nabla v_j^1\|_{P_j^1}^2 \lesssim \|h^{-\frac{1}{2}} \overline{\llbracket u_h \rrbracket}_{\Gamma_j}^{\Gamma_j}\|^2. \quad (3.14)$$

Using the trace inequality (3.1), Lemma 2.3 can be extended to

$$\sum_{j=1}^{N_p} \left\| \gamma^{\frac{1}{2}} \overline{\llbracket u_h \rrbracket}_{\Gamma_j}^{\Gamma_j} \right\|_{\Gamma_j} \geq \sum_{j=1}^{N_p} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket \right\|_{\Gamma_j} - C\omega_1 \sum_{j=1}^{N_p} \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1} - C\omega_2 \sum_{j=1}^{N_p} \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}. \quad (3.15)$$

We now have

$$\omega_1 = \frac{\mu_2}{\mu_1 + \mu_2}, \quad \omega_2 = \frac{\mu_1}{\mu_1 + \mu_2}, \quad \gamma = \frac{\mu_1 \mu_2}{h(\mu_1 + \mu_2)}.$$

3.2. Finite element formulation. To handle this problem we use the fictitious domain method on both subdomains Ω_1 and Ω_2 . We assume $\mu_1 \leq \mu_2$. Using the penalty free Nitsche's method, the finite element formulation for the problem (1.1) is written as : find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) + J_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h^k, \quad (3.16)$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{i=1}^2 (\mu_i \nabla u_h^i, \nabla v_h^i)_{\Omega_i} - \langle \{\mu \nabla u_h \cdot n\}, \llbracket v_h \rrbracket \rangle_{\Gamma} + \langle \{\mu \nabla v_h \cdot n\}, \llbracket u_h \rrbracket \rangle_{\Gamma}, \\ L_h(v_h) &= \sum_{i=1}^2 (f_i, v_h^i)_{\Omega_i}. \end{aligned}$$

The operator J_h is the ghost penalty [6], defined such that $J_h(u_h, v_h) = J_h^1(u_h^1, v_h^1) + J_h^2(u_h^2, v_h^2)$ with

$$J_h^1(u_h^1, v_h^1) = \gamma_g \sum_{F \in \mathcal{F}_G^1} \sum_{l=1}^k \langle \mu_1 h^{2l-1} \llbracket D_{n_F}^l u_h^1 \rrbracket, \llbracket D_{n_F}^l v_h^1 \rrbracket \rangle_F,$$

$$J_h^2(u_h^2, v_h^2) = \gamma_g \sum_{F \in \mathcal{F}_G^2} \sum_{l=1}^k \langle \mu_2 h^{2l-1} \llbracket D_{n_F}^l u_h^2 \rrbracket, \llbracket D_{n_F}^l v_h^2 \rrbracket \rangle_F.$$

This penalization ensures the stability in the case of small cut elements that might cause a dramatic growth of the condition number. The sets \mathcal{F}_G^i for $i = 1, 2$ are often called the interior facets

$$\mathcal{F}_G^1 := \{F \in G_h \mid F \cap \Omega^1 \neq \emptyset\} \quad , \quad \mathcal{F}_G^2 := \{F \in G_h \mid F \cap \Omega^2 \neq \emptyset\} .$$

$D_{n_F}^l$ is the partial derivative of order l in the direction n_F . Figure 6 shows an example of the sets \mathcal{F}_G^1 and \mathcal{F}_G^2 for a small part of the boundary. From [20] we have the estimate

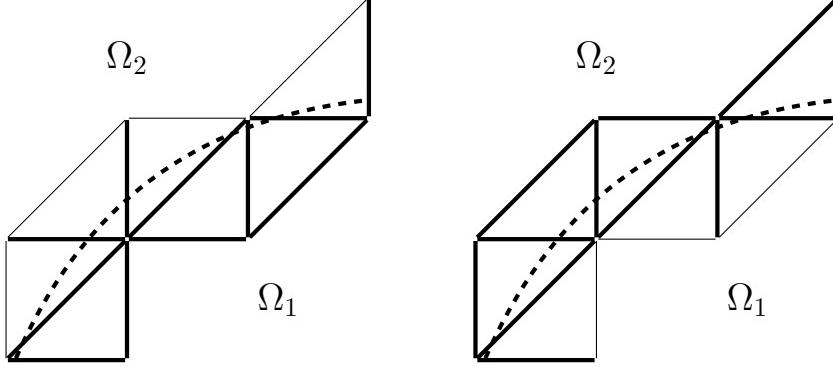


FIG. 6. Zoom of Figure 4, the dashed line is the boundary Γ , the bold facets are in the set \mathcal{F}_G^i , left \mathcal{F}_G^1 , right \mathcal{F}_G^2 .

$$\left\| \mu_i^{\frac{1}{2}} \nabla v_h^i \right\|_{\Omega_i^*}^2 \lesssim \left\| \mu_i^{\frac{1}{2}} \nabla v_h^i \right\|_{\Omega_i}^2 + J_h^i(v_h^i, v_h^i) \lesssim \left\| \mu_i^{\frac{1}{2}} \nabla v_h^i \right\|_{\Omega_i^*}^2 . \quad (3.17)$$

3.3. Inf-sup stability.

We define the norm

$$\|w\|_*^2 := \sum_{i=1}^2 \left\| \mu_i^{\frac{1}{2}} \nabla w^i \right\|_{\Omega_i^*}^2 + \left\| \gamma^{\frac{1}{2}} [w] \right\|_{\Gamma}^2$$

LEMMA 3.1. For $u_h, v_h \in V_h^k$ with $v_h = u_h + v_\Gamma^1$, v_Γ^1 defined by equations (3.9) and (3.10), there exists a positive constant β_0 such that the following inequality is true

$$\beta_0 \|u_h\|_*^2 \leq A_h(u_h, v_h) + J_h(u_h, v_h)$$

Proof. Using (3.17) we can write

$$A_h(u_h, u_h) + J_h(u_h, u_h) = \sum_{i=1}^2 (\mu_i \nabla u_h^i, \nabla u_h^i)_{\Omega_i} + J_h(u_h, u_h) \gtrsim \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{\Omega_1^*}^2 + \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{\Omega_2^*}^2$$

$$\begin{aligned} (A_h + J_h)(u_h, v_j^1) &= (\mu_1 \nabla u_h^1, \nabla v_j^1)_{P_j^1 \cap \Omega_1} - \langle \{\mu \nabla u_h \cdot n\}, v_j^1 \rangle_{\Gamma_j} \\ &\quad + \omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, [u_h] \rangle_{\Gamma_j} + J_h(u_h, v_j^1). \end{aligned}$$

Using the proof of Lemma 2.4 with (3.1) (3.14)

$$\begin{aligned} (\mu_1 \nabla u_h^1, \alpha \nabla v_j^1)_{P_j^1 \cap \Omega_1} + \alpha J_h(u_h, v_j^1) &\geq - \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1 \cap \Omega_1} \alpha \mu_1^{\frac{1}{2}} \| \nabla v_j^1 \|_{P_j^1 \cap \Omega_1} \\ &\quad - J_h(u_h, u_h)^{\frac{1}{2}} \alpha J_h(v_j^1, v_j^1)^{\frac{1}{2}} \\ &\gtrsim -\epsilon \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 - \frac{C\alpha^2 \mu_1}{4\epsilon} \| \nabla v_j^1 \|_{P_j^1}^2 \\ &\gtrsim -\epsilon \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 - \frac{C\alpha^2}{4\epsilon} \left(1 + \frac{\mu_1}{\mu_2} \right) \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{\Gamma_j} \right\|_{\Gamma_j}^2, \end{aligned}$$

Using the trace and inverse inequalities, (3.13) and (3.14) we have

$$\langle \{\mu \nabla u_h \cdot n\}, \alpha v_j^1 \rangle_{\Gamma_j} \leq \frac{C\alpha^2}{2\epsilon} \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{\Gamma_j} \right\|_{\Gamma_j}^2 + \epsilon \omega_1 \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 + \epsilon \omega_2 \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2.$$

Using (3.11), (3.14), (3.12) the trace and the inverse inequalities we obtain

$$\begin{aligned} \alpha \omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, \llbracket u_h \rrbracket \rangle_{\Gamma_j} &\geq -\epsilon \omega_1 \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 - \epsilon \omega_2 \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2 \\ &\quad + \alpha \left(1 - \frac{C\alpha}{2\epsilon} \right) \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket^{\Gamma_j} \right\|_{\Gamma_j}^2. \end{aligned}$$

We now have the lowerbound

$$\begin{aligned} A_h(u_h, v_h) &\geq \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{\Omega_1^* \setminus P^1}^2 + \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{\Omega_2^* \setminus P^2}^2 + \sum_{j=1}^{N_p} \left[(C_a - \omega_1 C C_c) \left\| \mu_1^{\frac{1}{2}} \nabla u_h^1 \right\|_{P_j^1}^2 \right. \\ &\quad \left. + (C_b - \omega_2 C C_c) \left\| \mu_2^{\frac{1}{2}} \nabla u_h^2 \right\|_{P_j^2}^2 + C_c \left\| \gamma^{\frac{1}{2}} \llbracket u_h \rrbracket \right\|_{\Gamma_j}^2 \right], \end{aligned}$$

with the constants

$$\begin{aligned} C_a &= 1 - \epsilon (2\omega_1 + 1), \\ C_b &= 1 - 2\epsilon \omega_2, \\ C_c &= \alpha \left(1 - \alpha \frac{C}{4\epsilon} \left(5 + \frac{\mu_1}{\mu_2} \right) \right). \end{aligned}$$

All terms are positive for $\epsilon = \min \left[\frac{1}{2(2\omega_1+1)}, \frac{1}{4\omega_2} \right]$ and $\alpha = \min \left[\frac{4\epsilon \omega_1}{C(4\omega_1+1)}, \frac{1}{2C} \right]$. \square

THEOREM 3.2. *There exists a positive constant $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta \llbracket u_h \rrbracket_* \leq \sup_{v_h \in V_h^k} \frac{A_h(u_h, v_h) + J_h(u_h, v_h)}{\llbracket v_h \rrbracket_*}.$$

Proof. Same as proof of Theorem 2.5. \square

3.4. A priori error estimate.

We have the following consistency relation

LEMMA 3.3. *If $u \in H^2(\Omega_1) \times H^2(\Omega_2)$ is the solution of (1.1) and $u_h \in V_h^k$ the solution of (3.16) the following property holds*

$$A_h(u - u_h, v_h) - J_h(u_h, v_h) = 0, \quad \forall v_h \in V_h^k.$$

Let us introduce the norm

$$\|w\|_* = \|w\|_* + \left\| \mu_1^{\frac{1}{2}} h^{\frac{1}{2}} \nabla w^1 \cdot n \right\|_\Gamma + \left\| \mu_2^{\frac{1}{2}} h^{\frac{1}{2}} \nabla w^2 \cdot n \right\|_\Gamma.$$

LEMMA 3.4. *Let $w \in H^2(\Omega_1) \times H^2(\Omega_2) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant M such that the bilinear form $A_h(\cdot, \cdot)$ has the property*

$$A_h(w, v_h) \leq M \|w\|_* \|v_h\|_*.$$

Proof. Same as Lemma 2.7 with $h_1 = h_2 = h$. \square

LEMMA 3.5. *Let $u \in H^2(\Omega_1) \times H^2(\Omega_2)$ and π_h the interpolation operator defined by (3.3). For all $v_h \in V_h^k$ the following inequality holds*

$$J_h(\pi_h u, v_h) \leq C_J h^k \|u\|_{H^{k+1}(\Omega)} \|v_h\|_*,$$

with $O(C_J) = O(1)$.

Proof. The continuity of u gives $J_h(\mathbb{E}u, v_h) = 0$. Using the definition of the interpolant π_h (3.3), the inverse inequality (2.2), the interpolation estimate (3.6) and the continuity of \mathbb{E}

$$J_h(\pi_h u, v_h) = J_h(\pi_h^* \mathbb{E}u - \mathbb{E}u, v_h) \lesssim h^k |\mathbb{E}u|_{H^{k+1}(\Omega_h)} \|\nabla v_h\|_{\Omega_h} \lesssim h^k \|u\|_{H^{k+1}(\Omega)} \|v_h\|_*.$$

\square

PROPOSITION 3.6. *If $u \in H^{k+1}(\Omega_1) \times H^{k+1}(\Omega_2)$ is the solution of (1.1) and $u_h \in V_h^k$ the solution of (3.16), then there holds*

$$\|u - u_h\| \leq C_{f\mu} h^k |u|_{H^{k+1}(\Omega)}.$$

with $C_{f\mu}$ is a positive constant that depends on μ and the mesh geometry.

Proof. Using the orthogonality relation of Lemma 3.3 we can write

$$A_h(u_h - \pi_h u, v_h) + J_h(u_h - \pi_h u, v_h) = A_h(u - \pi_h u, v_h) - J_h(\pi_h u, v_h),$$

applying this property with Theorem 3.2 and Lemma 3.4 we get

$$\beta \|u_h - \pi_h u\|_* \leq \frac{A_h(u - \pi_h u, v_h) - J_h(\pi_h u, v_h)}{\|v_h\|_*} \leq M \|u - \pi_h u\|_* - \frac{J_h(\pi_h u, v_h)}{\|v_h\|_*}.$$

Using (3.4) we can write

$$\|u - \pi_h u\| \lesssim \|u - \pi_h u\|_* \lesssim \mu_1^{\frac{1}{2}} h^k |u^1|_{H^{k+1}(\Omega_1)} + \mu_2^{\frac{1}{2}} h^k |u^2|_{H^{k+1}(\Omega_2)}.$$

It is straightforward to observe that for any $v_h \in V_h^k$ we have $\|v_h\| \lesssim \|v_h\|_*$, using this result and the triangle inequality we have

$$\begin{aligned} \|u - u_h\| &\leq \|u - \pi_h u\| + \|u_h - \pi_h u\|_* \\ &\leq \|u - \pi_h u\| + \frac{1}{\beta} \left(M \|u - \pi_h u\|_* - \frac{J_h(\pi_h u, v_h)}{\|v_h\|_*} \right). \end{aligned}$$

Applying Lemma 3.5

$$\|u - u_h\| \lesssim \left(1 + \frac{M+1}{\beta} \right) h^k \|u\|_{H^{k+1}(\Omega)}. \quad \square$$

LEMMA 3.7. Let $u \in H^{k+1}(\Omega_1) \times H^{k+1}(\Omega_2)$ be the solution of (1.1) and u_h the solution of (3.16), then

$$\|u - u_h\|_{\Omega} \leq C'_{f\mu} h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)}$$

with $C'_{f\mu}$ is a positive constant that depends on μ and the mesh geometry.

Proof. Same proof as Lemma 2.9 considering the new unfitted framework. \square

4. Numerical verifications. In this section we verify numerically the convergences proven theoretically. For each case studied the domain considered is the unit square separated in two subdomains as it is shown in Figure 1. We use a manufactured solution in order to test the precision and determine the slopes of convergence. The manufactured solution that has been considered in this case

$$u = \exp(xy)\sin(\pi x)\sin(\pi y).$$

For fitted and unfitted domain decomposition we consider $\mu_1 = 1$ and we test a range of value for μ_2 .

4.1. Fitted domain decomposition. The package FreeFem++ [17] is used to implement this case. We choose different values of the ratio h_1/h_2 and observe the L^2 and H^1 -error for each configuration.

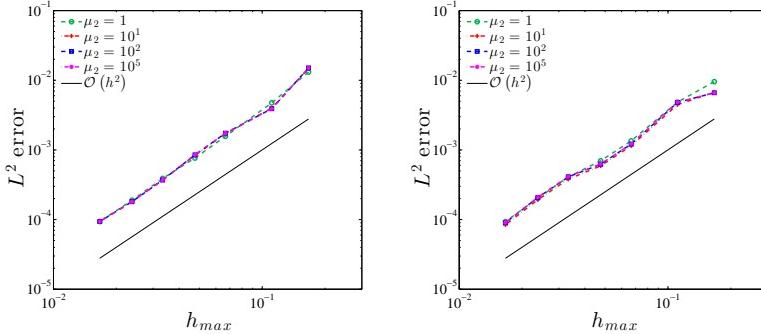


FIG. 7. Piecewise affine approximation, $\mu_1 = 1$, left $\frac{h_1}{h_2} = 1$, right $\frac{h_1}{h_2} = \frac{3}{5}$.

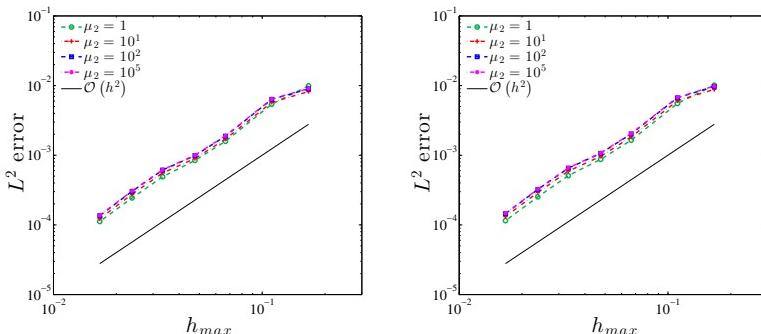


FIG. 8. Piecewise affine approximation, $\mu_1 = 1$, left $\frac{h_1}{h_2} = \frac{3}{10}$, right $\frac{h_1}{h_2} = \frac{1}{5}$.

Figure 7 and 8 shows a convergence of order $\mathcal{O}(h^2)$ for the L^2 -error, this is a super convergence of order $\mathcal{O}(h^{\frac{1}{2}})$ compared to the theoretical result. This super convergence has been observed for linear elasticity with the penalty free Nitsche's method in [4]. Comparing all four graphs we observe that as the ratio h_1/h_2 becomes smaller the constant C'_μ becomes slightly bigger when μ_2 grows.

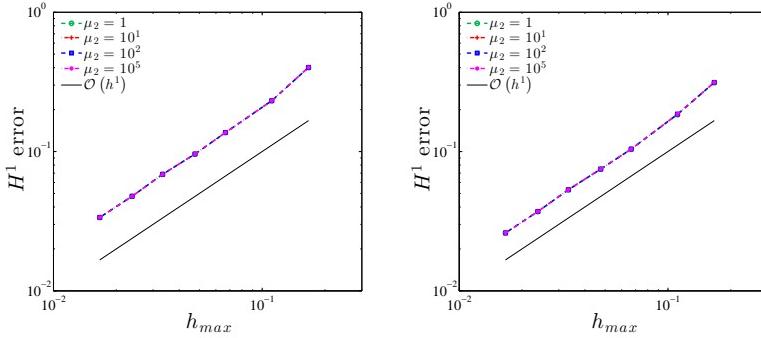


FIG. 9. Piecewise affine approximation, $\mu_1 = 1$, left $\frac{h_1}{h_2} = 1$, right $\frac{h_1}{h_2} = \frac{3}{5}$.

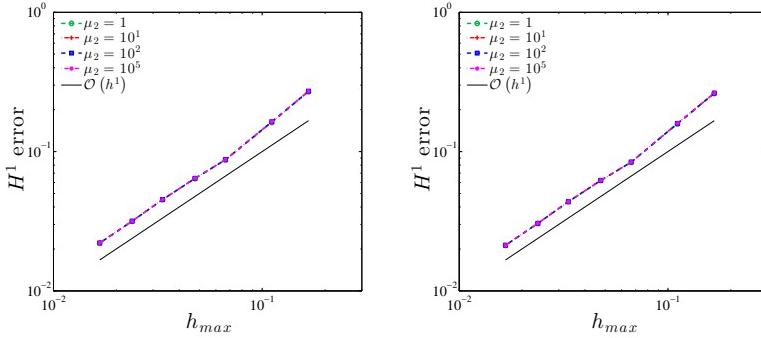
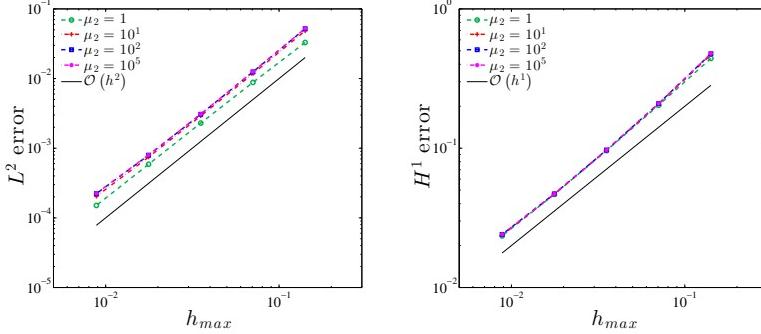


FIG. 10. Piecewise affine approximation, $\mu_1 = 1$, left $\frac{h_1}{h_2} = \frac{3}{10}$, right $\frac{h_1}{h_2} = \frac{1}{5}$.

We note that in all the cases the slope of convergence that has been proven theoretically is observed as it is shown in Figures 9 and 10. In fact the affine approximation considered gives slopes of convergence of order $\mathcal{O}(h)$ which is what has been shown theoretically. For $h_1/h_2 = 1$ the meshsize are the same on both sides of Γ , in this case the influence of μ_2 is negligible, the error remains the same for every value of μ_2 considered. By considering the ratio h_1/h_2 smaller, the nonconformity of the meshes on both size of gamma gets bigger but it has a very small impact on the error for the three nonconforming cases considered.

4.2. Unitted domain decomposition. The package FEniCS [19] and the library CutFEM [8] have been used for these computations.

FIG. 11. Piecewise affine approximation, unfitted case, $\gamma_g = 0.001$.

The same super convergence is observed as in the fitted case for the L^2 -error. The case $\mu_2 = 1$ seems to have a constant $C'_{f\mu}$ slightly smaller than the other cases. The H^1 -error shows the same convergence as shown in the theory, once again, the difference between each case is negligible.

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